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AN INVESTIGATION OF CONFIDENCE LIMITS ON SYSTEM RELIABILITY WHEN COMPONENT SAMPLE SIZES VARY

ROBERT THEODORE ISAACSON

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AN INVESTIGATION OF CONFIDENCE LIMITS

ON SYSTEM RELIABILITY WHEN COMPONENT

SAMPLE SIZES VARY

by

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[redacted]

Submitted in partial fulfillment
for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

UNITED STATES NAVAL POSTGRADUATE SCHOOL
May 1966

NPS ARCHIVE
1966
SPARACINO, P.

ABSTRACT

The method of [1] for obtaining an approximate lower $100(1-\alpha)\%$ confidence limit on System Reliability is used, except, instead of a Gamma probability distribution, a Normal probability distribution is used as the underlying distribution for $\hat{S}$, the estimator of the negative natural logarithm of system reliability.

An investigation is made of the errors involved in the truncations and approximations used throughout the development. A continuity correction factor is developed, and confidence limits on system reliability, resulting from computer simulation, are presented both with and without the inclusion of this continuity correction factor.

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1. INTRODUCTION

An *ad hoc* procedure for obtaining an approximate confidence interval for system reliability has been developed, based upon Bernoulli test data from unequal samples of the various components of the system [1]. Integral to this procedure is an assumption that the unbiased estimator of the negative natural logarithm of system reliability may be approximated by the Gamma probability distribution. It was the purpose of this thesis to (1) investigate the errors in the above mentioned procedure (hereafter called the Woods-Borsting Method) resulting from truncation and approximation; (2) develop a different confidence interval, using the same *ad hoc* procedure, but assuming an underlying normal probability distribution, vice Gamma, on the unbiased estimator of the negative natural logarithm of system reliability; and (3) compare the confidence limits resulting from both procedures for various values of the parameters involved.

That the investigation which follows was ever undertaken is primarily justified by the results obtained under the Gamma distribution assumption. As will be explained below, the α^{th} percentile of the simulated distribution of the lower confidence limit, call it $R_{S,L(\alpha)}$, and true system reliability, R_S, should be the same; hence, the absolute value of their difference can be used as a measure of the accuracy of the Woods-Borsting Method. In one case the magnitude of this difference was .18

and the average difference was .036. The above figures, coupled with the realization by the author's of the need for error analysis, motivated the undertaking of this thesis.

Although any one of several probability laws could have been chosen to supplant the Gamma distribution used in the Woods-Borsting Method, this investigation is limited to only one, the normal distribution. If we write the negative natural logarithm of system reliability as $-\ln R_S$ and define this quantity, for simplicity of notation, to be identically equal to S, then it will be seen below that two sources of error are directly related to S. The first is an error of truncation of an infinite series expansion of $\ln R_S$. The second is an error of approximation in the expression for Var $(\hat{S})$, where $\hat{S}$ is the unbiased estimator for S.

Error analysis is performed on the former, but not on the latter, because it is felt that, due to the complexity of the expression dropped in the approximation, this would, in itself, be subject matter sufficient for a thesis.

In the following sections, the necessary background is first presented, which includes an explanation of the Woods-Borsting Method, the investigation of the two above mentioned errors, the formal introduction of the normal distribution assumption, and the mathematical development of a new confidence interval based on this assumption of normality.

In Section 3, the lower confidence limit, taken from the confidence interval, is seen to a function of random variables,

and hence has some underlying probability distribution. This probability distribution is simulated by digital computer, the shape of the distribution, and the reasons therefor , are discussed, the results are presented in tabled form, and compared to the results of the same simulation of the Woods-Borsting Method. It will be seen that the results are in need of improvement, leading to Section 4.

In Section 4, a continuity correction factor is developed, necessitated by the fact that $\hat{S}$, assumed to be a continuous random variable, can, in reality, take on only discrete values. This continuity correction factor is then incorporated in the confidence limit, the distribution of this new random variable simulated, the results tabulated, and comparison made to both the results of Section 3 and the Woods-Borsting Method.

In the last section, a summary comparison is made, in a condensed tabular form and in the text, of the results of the preceding two sections, the Woods-Borsting Method, and a well-known method of estimating system reliability when all component sample sizes are the same, based on the Poisson approximation to the Binomial distribution. Conclusions are made as to the acceptability of the Normal assumption over the Gamma, and to the extent to which the addition of the continuity correction factor increases the accuracy.

Ever since automation and advancing technology began to increase the complexity of various equipments, particularly military weapons systems, the purchasers, and consequently the

producers, of these equipments have become increasingly more aware of the necessity for predicting the overall reliability of these equipments. Many problems in the mathematical theory of reliability have been solved, but one that has not is that of obtaining a confidence interval on system reliability when the reliability of the system is to be computed as the product of the various component reliabilities, and the sizes of the test samples taken of the various components are not all equal. If component sample sizes are all equal, there exists a well-known method of obtaining a confidence interval on system reliability, based on the Poisson approximation to the Binomial, as explained in [1], p. 2, and [3], pp. 218-219. However, when one considers that components of complex systems are being built by different sub-contractors in different geographical locations, that these components are not mass produced, rather only a relatively small quantity of each are ever manufactured, and that samples are drawn for reliability testing before the whole system is ever assembled, then the only conclusion is that sample sizes of components will rarely be the same, except by accident, not design. Hence, there is a definite need for a method of obtaining a confidence interval on system reliability, based upon varying sample sizes of components. Using the work done in [1] as a foundation, this thesis is an attempt to move toward the solution of this problem.

2. BACKGROUND

In this section the Woods-Borsting Method is explained and the mathematical development shown, errors in the Woods-Borsting Method due to truncation and approximation are pointed out and investigated, the assumption of an underlying normal probability distribution for $\hat{S}$ is introduced, and the lower $100(1-\alpha)\%$ confidence interval (where α = P[type I error]) based on the normal assumption is developed.

EXPLANATION OF WOODS-BORSTING METHOD

In a system with k components connected in logical series, the true system reliability, R_S, may be expressed as

$$R_S = \prod_{i=1}^{k} p_i \tag{1}$$

where p_i is the true reliability of the i^{th} component, whether the component be continuous operating or cycling type. Component reliability may be expressed in terms of unreliability, q_i.

$$p_i = 1 - q_i \tag{2}$$

and hence

$$R_S = \prod_{i=1}^{k} (1 - q_i) \tag{3}$$

Define

$$S \equiv -\ln R_S = -\sum_{i=1}^{k} \ln (1 - q_i) \tag{4}$$

and, expanding the natural logarithm in an infinite series

$$S = -\sum_{i=1}^{k} \left\{ (-q_i) - \frac{1}{2}(-q_i)^2 + \frac{1}{3}(-q_i)^3 - \cdots \right\}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{\infty} \frac{q_i^j}{j} \qquad (5)$$

which may be approximated by the first two terms of the series, called T_i for ease of expression.

$$S \doteq \sum_{i=1}^{k} q_i + \frac{q_i^2}{2} \equiv \sum_{i=1}^{k} T_i \qquad (6)$$

It is shown in Section 2 that the error due to truncation is quite small; in fact,

$$S \leq \sum_{i=1}^{k} T_i + \sum_{i=1}^{k} \frac{q_i^3}{3(1 - q_i)} \qquad (7)$$

It can be shown (see Appendix I) that an unbiased estimator for T_i, call it $\hat{T}_i$, is

$$\hat{T}_i = a_i \hat{q}_i + b_i \frac{\hat{q}_i^2}{2} \qquad (8)$$

where

$$a_i = \frac{2n_i - 3}{2(n_i - 1)} \qquad (9)$$

$$b_i = \frac{n_i}{n_i - 1} \qquad (10)$$

$$\hat{q}_i = \frac{f_i}{n_i} \qquad (11)$$

n_i being the number of units of component i that lived out their assigned mission time during a test, and f_i the number of failures of type i observed during the test. Therefore

$$\hat{S} = \sum_{i=1}^{k} \hat{T}_i \tag{12}$$

is an unbiased estimator of S. An approximate value for the variance of $\hat{S}$ can be shown to be (see Appendix II)

$$\mathrm{Var}(\hat{S}) = \sum_{i=1}^{k} \mathrm{Var}(\hat{T}_i) \doteq \sum_{i=1}^{k} \frac{T_i}{n_i} \tag{13}$$

Thus, we have the first two sample moments of the random variable $\hat{S}$.

Next, a two-parameter Gamma distribution is fitted to $\hat{S}$ by the method of moments. If the density of $\hat{S}$ is taken to be in the form

$$f_{\hat{S}}(x;r,\theta) = \begin{cases} \dfrac{x^{r-1} \exp(-x/\theta)}{\Gamma(r)\,\theta^r}, & x > 0,\ r > 0,\ \theta > 0 \\ 0 & \text{otherwise} \end{cases}$$

then

$$E[\hat{S}] = \theta r = S$$

$$\mathrm{Var}(\hat{S}) = \theta^2 r = \sum_{i=1}^{k} \frac{T_i}{n_i}$$

Solving these equations for r and θ

$$r = \left(\sum_{i=1}^{k} T_i\right)^2 \Big/ \left(\sum_{i=1}^{k} \frac{T_i}{n_i}\right)$$

$$\theta = \sum_{i=1}^{k} \frac{T_i}{n_i} \Big/ \sum_{i=1}^{k} T_i$$

and thus an estimator $\hat{r}$, for r is taken as

$$\hat{r} = \left(\sum_{i=1}^{k} \hat{T}_i\right)^2 \Big/ \left(\sum_{i=1}^{k} \frac{T_i}{n_i}\right)$$

Since $\hat{S}$ is assumed to be Gamma-distributed, $2\hat{S}/\theta$ is distributed approximately χ^2_{2r}. Thus

$$1 - \alpha = P\left[\frac{2\hat{S}}{\theta} \geq \chi^2_{1-\alpha,\, 2r}\right]$$

$$= P\left[\theta r \leq \hat{S}^{2r} \Big/ \chi^2_{1-\alpha,\, 2r}\right]$$

$$= P\left[-\ln R_S \leq \frac{2r\,\hat{S}}{\chi^2_{1-\alpha,\, 2r}}\right]$$

Where $\chi^2_{1-\alpha,2r}$ is the α^{th} percentile point in χ^2_{2r} distribution.

> If $r \geq 3$ and α is small, the quantity $2r/\chi^2_{1-\alpha,2r}$ is almost a constant with respect to r. ...consequently, the random variable $2\hat{r}/\chi^2_{1-\alpha,2\hat{r}}$ will have small variance ... Thus, from this last equation... [1]

$$1 - \alpha \doteq P\left[-\ln R_S \leq \hat{S}\,[2\hat{r}] \Big/ \chi^2_{1-\alpha,[2\hat{r}]}\right]$$

$$= P\left[R_S \geq \exp\left(-\hat{S}\,[2\hat{r}] \Big/ \chi^2_{1-\alpha,[2\hat{r}]}\right)\right]$$

where $[2\hat{r}]$ denotes the smallest integer greater than or equal to $2\hat{r}$. The right-hand side of the probability statement is therefore the lower confidence limit of the Woods-Borsting Method, call it $R_{S_{L.C.}}$, and

$$R_{S_{L.C}} = \exp\left(-\hat{S}[2\hat{r}] \Big/ \chi^2_{1-\alpha,[2\hat{r}]}\right)$$

ANALYSIS OF THE ERROR DUE TO TRUNCATION

In the transition from (5) to (6) an infinite series was truncated to its first two terms. Examine the inner summation of (5)

$$\sum_{j=1}^{\infty} \frac{q_i^j}{j} = q_i + \frac{q_i^2}{2} + \sum_{j=3}^{\infty} \frac{q_i^j}{j} \tag{14}$$

$$= T_i + R_2$$

where R_2 is the remainder of the series after two terms have been written out. If we test the n^{th} and $(n + 1)^{st}$ terms of this series by the ratio test,

$$\frac{q_i^{n+1} \Big/ n+1}{q_i^n \Big/ n} = \frac{nq_i}{n+1} \tag{15}$$

we see that this expression is always less than $q_i \leq 1$, which implies the series is convergent. Then by Theorem 24, p. 328 of [2].

$$\lim_{n \to \infty} \left| \frac{nq_i}{n+1} \right| = q_i \tag{16}$$

which implies that

$$R_n \leq q_i^{n+1} \Big/ (n+1)(1-r), \quad r \geq q_i \tag{17}$$

Since $r \geq q_i$, let us choose $r = q_i \leq 1$ so as to make R_n as small as possible

$$R_n \leq q_i^{n+1} \Big/ (n+1)(1-q_i) \tag{18}$$

and evaluating at n = 2

$$R_2 \leq q_i^3 \Big/ 3(1-q_i) \tag{19}$$

Since $q_i \ll 1$ in most cases of interest, choose $q_i = .15$, for example, which is about the largest value it will ever take on, and we see that $R_2 \leq 1.4 \times 10^{-3}$. Substituting equation (19) into (14),

$$\sum_{j=1}^{\infty} \frac{q_i^j}{j} \leq T_i + q_i^3 \Big/ 3(1-q_i) \tag{20}$$

and substituting (20) into (5)

$$S \leq \sum_{i=1}^{k} (T_i + q_i^3 \Big/ 3(1-q_i)) \tag{21}$$

which upon distributing the summation sign becomes equation (7). For $q_i = .05$, still quite a large value for q_i to take on, and k = 15

$$S \leq \sum_{i=1}^{k} T_i + 6 \times 10^{-5} , \tag{21a}$$

so the approximation

$$S \doteq \sum_{i=1}^{k} T_i \tag{6}$$

is quite good.

INVESTIGATION OF ERROR DUE TO APPROXIMATION IN Var ($\hat{S}$)

In Appendix II the Variance of the random variable $\hat{S}$ is computed, and it is seen that the value used in (13) above differs

from the true variance by a factor

$$\sum_{i=1}^{k}\left(\frac{q_i^{\,2}}{2(n_i-1)} \left\{ 1 + \frac{3q_i^{\,2}}{n_i} - 2q_i^{\,2} - 2q_i \right\} \right) \tag{22}$$

Upon embarking upon an investigation of the size of this term it is quickly seen that it is a task which, due to the amount of work involved, is beyond the scope of this thesis. A few general comments on this term are in order, however.

Being a function of both the q_i and n_i, equation (22) may, for certain combinations of these parameters, actually be less than zero and reduce the variance of $\hat{S}$. In order to determine when this occurs, it would be necessary to use a digital computer to compute and plot for each set of n_i, $i=1, \cdots, k$, a curve over the range $0 \le q_i \le 1$; and, similarly, to compute and plot for each set of q_i, $i=1, \cdots, k$, a curve over the range of interest of n_i, $2 \le n_i \le N$, where N might be 150.

If the summand of (22) is plotted as a function of a single q_i, with the n_i held fixed, the curve is seen to be bell-shaped over the range $0 \le q \le 1$, but within the realistic range that q_i may be expected to take on, $0 \le q_i \le .15$, (22) is a non-decreasing function, and strictly increasing unless a point of inflection occurs.

If the summand of (22) is plotted as a function of a single n_i, where n_i is of course integer-valued, it is seen to be a monotone-decreasing step-function, asymptotic to the n_i axis.

As a crude bound, on the size of the error, consider the summand of (22)

$$\frac{q_i^2}{2(n_i-1)} \left\{ 1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i \right\} \tag{23}$$

$$< \frac{q_i^2}{2n_i} \left\{ 1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i \right\} \tag{24}$$

$$< \frac{q_i^2}{2n_i} \left\{ 1 + \frac{3q_i^2}{n_i} \right\} \tag{25}$$

$$< \frac{1}{2n_i} \left\{ 1 + \frac{3}{n_i} \right\} \tag{26}$$

which implies that

$$\sum_{i=1}^{k} \left(\frac{q_i^2}{2(n_i-1)} \left\{ 1 + \frac{3q_i^2}{n_i} - 2q_i - 2q_i \right\} \right) < \sum_{i=1}^{k} \frac{1}{2n_i} \left\{ 1 + \frac{3}{n_i} \right\} \tag{27}$$

which is obviously not a least upper bound, but an example is informative. Let n = 20 for all i, and k = 15

$$\sum_{i=1}^{k} \frac{1}{2n_i} \left\{ 1 + \frac{3}{n_i} \right\} \doteq .4 \tag{28}$$

which admittedly is of an undesirable magnitude for an omitted term, but (22) is undoubtedly much less than .4.

INTRODUCTION OF THE NORMAL DISTRIBUTION FIT FOR $\hat{S}$ AND DEVELOPMENT OF THE ASSOCIATED CONFIDENCE INTERVAL

Now, instead of a two-parameter Gamma distribution as was used in [1], assume that the distribution of $\hat{S}$ can be fitted by a Normal distribution with parameters μ and σ^2. The probability density function of $\hat{S}$ is therefore

$$f_S(x;\mu,\sigma^2) = \begin{cases} \frac{1}{\sigma\sqrt{2\Pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), & \sigma > 0 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

Again using the method of moments,

$$E[\hat{S}] = \mu = S \equiv -\ln R_S$$

$$\mathrm{Var}(\hat{S}) = \sigma^2 = \sum_{i=1}^{k} \frac{T_i}{n_i} \tag{30}$$

Let

$$\hat{\sigma}^2 = \sum_{i=1}^{k} \frac{\hat{T}_i}{n_i} \tag{31}$$

Then

$$E[\hat{\sigma}^2] = E\left[\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}\right]$$

$$= \sum_{i=1}^{k} E\left[\frac{\hat{T}_i}{n_i}\right] \tag{32}$$

$$= \sum_{i=1}^{k} \frac{T_i}{n_i}$$

and $\hat{\sigma}^2$ is an unbiased estimator of σ^2.

Now, based upon the normality assumption on $\hat{S}$,

$$1 - \alpha = P\left[\frac{\hat{S}-S}{\sigma} \geq -K_\alpha\right] \tag{33}$$

Where K_α is the ordinate of the tabled standard normal distribution exceeded with probability α. Multiplying by -1, equation (33) becomes

$$1 - \alpha = P\left[\frac{S-\hat{S}}{\sigma} \leq K_\alpha\right] \tag{34}$$

which may be approximated by

$$1 - \alpha \doteq P\,[S \le \hat{S} + K_\alpha \hat{\sigma}\,] \tag{34a}$$

$$= P\,[S \le \hat{S}_{U(\alpha)}\,]$$

if $\hat{\sigma}$ is the square root of the unbiased estimator of σ^2. Since $\hat{\sigma}^2$ was seen to be unbiased, let

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}} \tag{35}$$

Putting in the values for $\hat{S}$ and $\hat{\sigma}$ in (34a)

$$1 - \alpha \doteq P\left[S \le \sum_{i=1}^{k} \hat{T}_i + K_\alpha \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}}\,\right]$$

$$= P\left[-\ln R_S \le \sum_{i=1}^{k} \hat{T}_i + K_\alpha \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}}\,\right] \tag{36}$$

$$= P\left[R_S \ge \exp\left(-\sum_{i=1}^{k} \hat{T}_i - K_\alpha \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}}\right)\right]$$

which is an approximate lower $100(1-\alpha)\%$ confidence interval on R_S. The associated lower confidence limit, call it $R_{S,L(\alpha)}$, is

$$R_{S,L(\alpha)} = \exp\left(-\sum_{i=1}^{k} \hat{T}_i - K_\alpha \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}}\right) \tag{37}$$

Substituting equations (8) through (11) in the above expression, the result is

$$R_{S,L(\alpha)} = \exp\left(-\sum_{i=1}^{k}\left(\frac{a_i f_i}{n_i} + \frac{b_i f_i^2}{2n_i^2}\right) - K_\alpha \sqrt{\sum_{i=1}^{k} \frac{1}{n_i}\left(\frac{a_i f_i}{n_i} + \frac{b_i f_i^2}{2n_i^2}\right)}\right) \tag{38}$$

which is a random variable whose distribution is unknown, although it is a function of the Binomial random variables f_i, $i = 1, \cdots, k$ and the parameters n_i and k. Thus, to evaluate the worth of this confidence limit it is necessary to simulate the distribution of this random variable.

3. SIMULATED DISTRIBUTION OF CONFIDENCE LIMIT AND RESULTS

In this section the digital computer simulation of the distribution of $\hat{S} + K_\alpha \hat{\sigma}$ is explained, as is the method of testing the accuracy of the results of this simulation. Several histograms of the computer-simulated distributions are presented in Figures 1 through 4, with a discussion of the shapes of these histograms. The results of the computer simulation are presented for ten different combinations of the input parameters in Table I and these results compared with the same cases of the Woods-Borsting Method in Table II.

SIMULATED DISTRIBUTION OF THE CONFIDENCE LIMIT AND MEASURE OF ACCURACY

In order to simulate the distribution of the confidence limit, the input parameters k, n_i, $i = 1, \cdots, k$, and $p_i = 1 - q_i$, $i = 1, \cdots, k$ are chosen and used as inputs to be computer program. The values chosen are given in Table I. Using true values, the following are computed in order that they may be compared with the simulated values:

$$R_S = \prod_{i=1}^{k} p_i \tag{1}$$

$$T_i = q_i + \frac{q_i^2}{2} \quad , \; i = 1, \cdots, k \tag{39}$$

$$S = \sum_{i=1}^{k} T_i \tag{6}$$

$$S = -\ln R_S \tag{4}$$

$$\sigma = \sqrt{\sum_{i=1}^{k} \frac{T_i}{n_i}} \tag{40}$$

Note that S is computed two different ways in order to see the effect of the truncation of the logarithmic series.

Next, the simulated values are generated by the program. A three digit random number is obtained using a uniform random number generator subroutine. If the random number is greater than p_i, a failure is "counted" by the computer; if the random number is less than or equal to p_i, no "count" is made. This random number generation is done n_i times for each p_i. Thus the number of failures counted divided by the number of units of that component becomes and estimator of unreliability for that component; i.e.,

$$\hat{q}_i = \frac{f_i}{n_i} \tag{11}$$

The whole process is repeated k times, giving an estimator for

each q_i. Several arithmetic operations are performed by the computer with these q_i resulting in computation and storage for further use of the following simulated values:

$$\hat{T}_i = a_i \hat{q}_i + \frac{b_i \hat{q}_i^2}{2} \tag{8}$$

$$\hat{S} = \sum_{i=1}^{k} \hat{T}_i \tag{12}$$

$$\hat{\sigma} = \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}} \tag{41}$$

In order to complete the simulation of the distribution, it was decided to make 500 replications of the above procedure. Thus 500 values each of (12) and (41) are computed and combined in the form

$$\hat{S}_{U(\alpha)} = \sum_{i=1}^{k} \hat{T}_i + K_\alpha \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i}{n_i}} \tag{42}$$

for each value of α. Three values, .05, .10, .20, were chosen for α. The 500 points of the distribution were then sorted, by size, by a separate subroutine. The distribution was printed out in its entirety in order that a frequency histogram could be plotted, and the α^{th} percentile of this distribution, call it A, was found by the computer. Then e^{-A} was computed and printed out for comparison with true system reliability, R_S.

Consider the underlying meaning of the probability statement

$$P[\, R_S \geq R_{S,L(\alpha)} \,] = 1 - \alpha \tag{43}$$

If the probability density function of $R_{S,L(\alpha)}$ is plotted on a coordinate system, and R_S is taken to be a point on the abscissa, then $(1 - \alpha)\%$ of the area under the curve lies above and $\alpha\,\%$ lies below R_S. Therefore, R_S should be equal to the α^{th} percentile of the distribution of $R_{S,L(\alpha)}$ which we have called $\exp(-A)$ above. Thus

$$\left| \, R_S - \exp(-A) \, \right| \tag{44}$$

is a measure of the accuracy of the procedure under investigation. This difference is shown in Table I for all cases simulated.

DISCUSSION OF SHAPE OF DISTRIBUTION

As will be seen in Table I, ten different combinations of the parameters k, n_i, and p_i were selected for computer simulation; for each of these ten combinations three values of α, .05, .10, and .20, were used, for a total of 30 different distributions simulated. Four of these 30 were chosen as illustrative of the effect the parameters have on the shape of the distribution, and are included as Figures 1 through 4. In all cases illustrated $\alpha = .20$ and $k = 15$, except Figure 2 where $k = 13$. All the figures utilize class intervals of width .025.

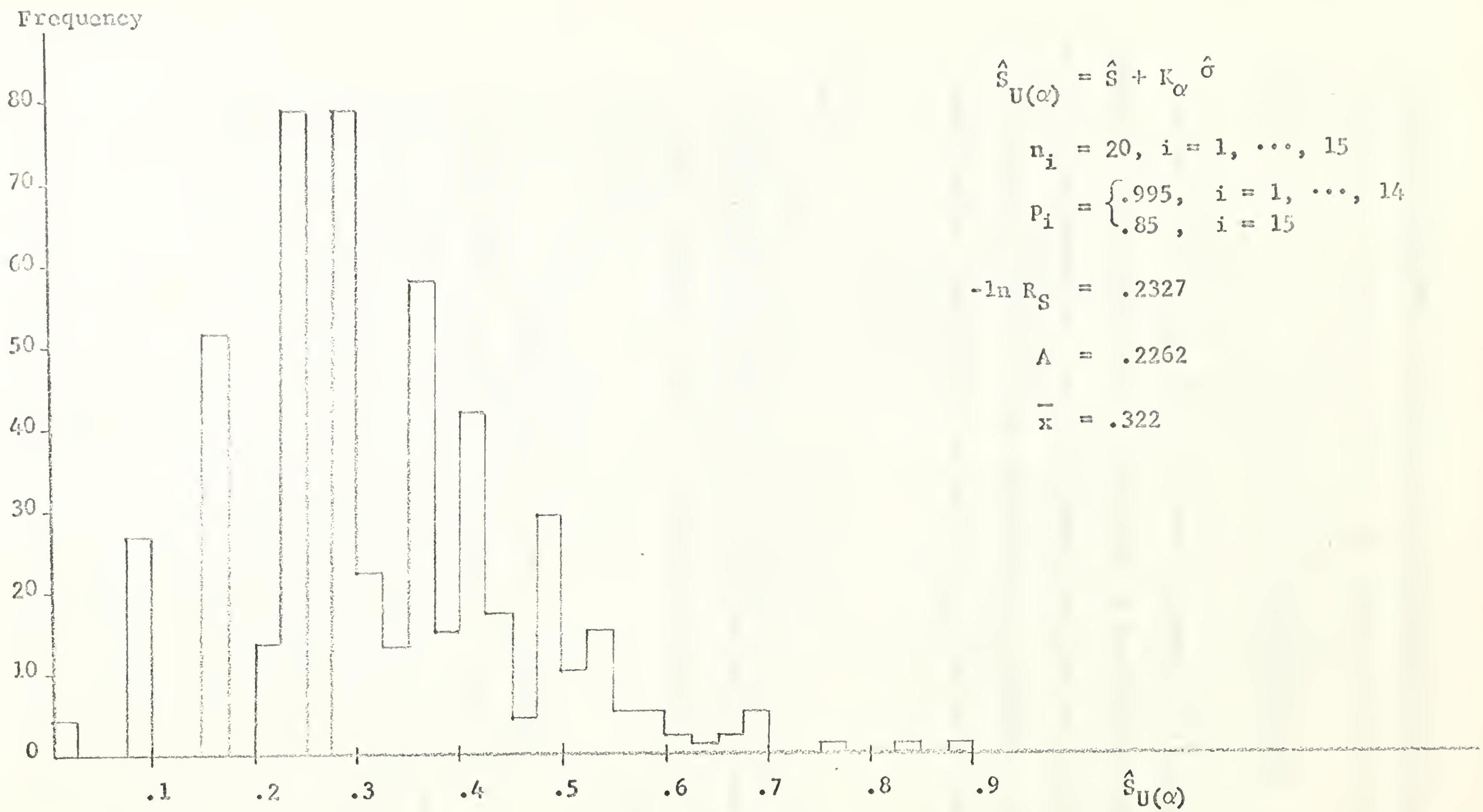


FIGURE 1

HISTOGRAM OF DISTRIBUTION FOR CASE 1, $\alpha = .20$

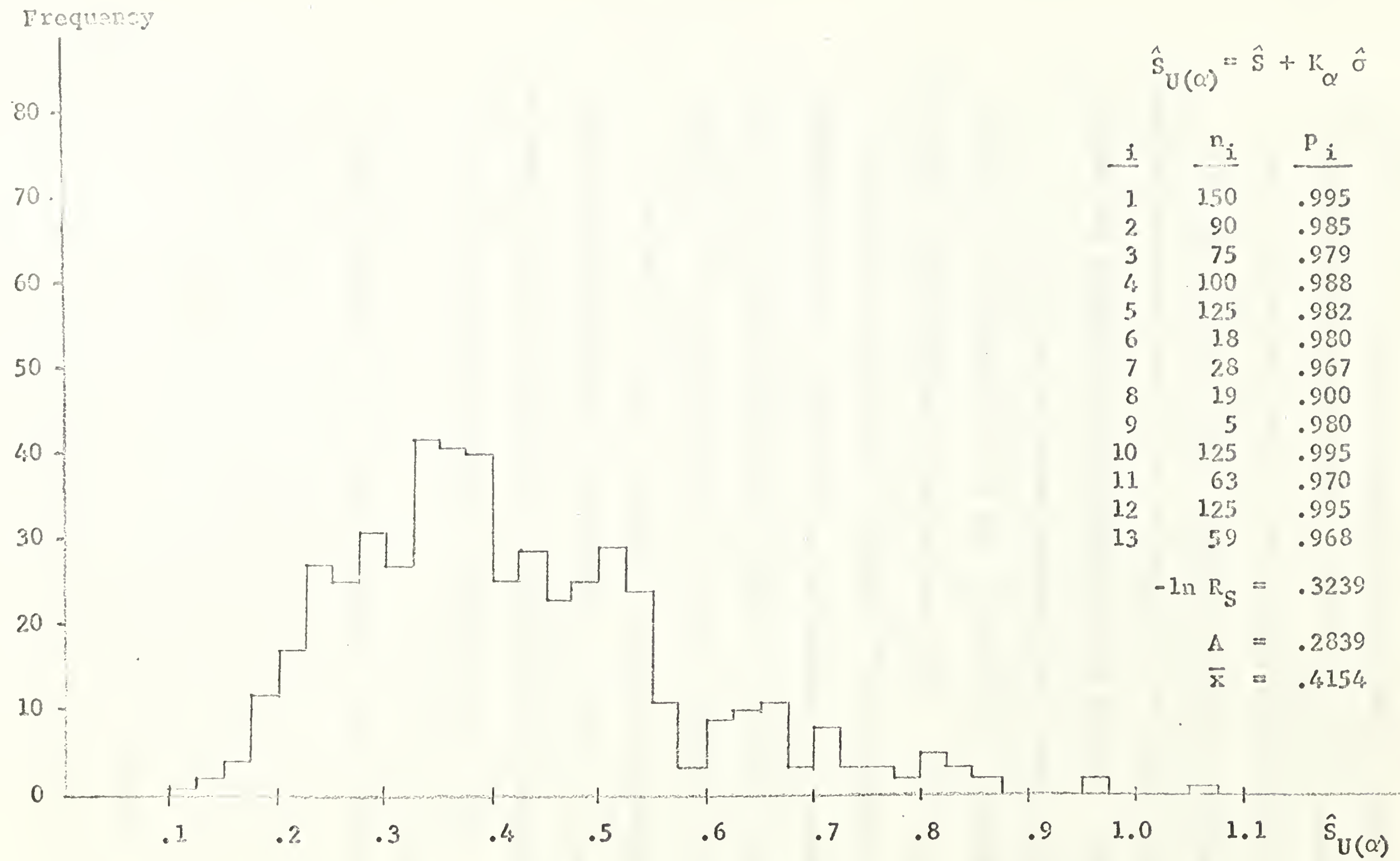
All four are skewed right in varying degree, having greater than 50% of the area below the mean.

In Figure 1, the n_i are all equal as are 14 of the 15 p_i. The remaining p_i, $p_{15} = .85$, is considerably lower than the rest. In this case, six of the 12 class intervals in the lower tertile have zero frequency, and these six are not adjacent, rather are interspersed with class intervals of high frequency. Overall the distribution is multimodal and and has a range of .8868. The standard deviation is .136.

In Figure 2, both the n_i and p_i vary greatly, $5 \le n_i \le 150$, $.900 \le p_i \le .995$. This distribution is unimodal, and, except for an area between the 87th and 96th percentiles, approximates a right-skewed bell-shaped curve. The range of the distribution is .9545, from .108 to 1.0625; the standard deviation is .153.

All p_i are equal, but the n_i vary greatly, $15 \le n_i \le 250$, in the next case, shown in Figure 3, The distribution is unimodal, and closely approximates a bell-shaped curve, with a slight degree of skewness to the right. Here is the smallest range, .3696, and the smallest standard deviation, .06, of all distributions simulated.

The last case displayed, Figure 4, has relatively small values for the p_i, .95 for the first 14, and .85 for the 15th. The n_i are all equal to 20. This distribution is erratic over its entire range, the class intervals alternating from low to high frequency. The range of the distribution is 1.4063, but it is interesting to note that the distribution is shifted much



$$\hat{S}_{U(\alpha)} = \hat{S} + K_\alpha \hat{\sigma}$$

i	n_i	P_i
1	150	.995
2	90	.985
3	75	.979
4	100	.988
5	125	.982
6	18	.980
7	28	.967
8	19	.900
9	5	.980
10	125	.995
11	63	.970
12	125	.995
13	59	.968

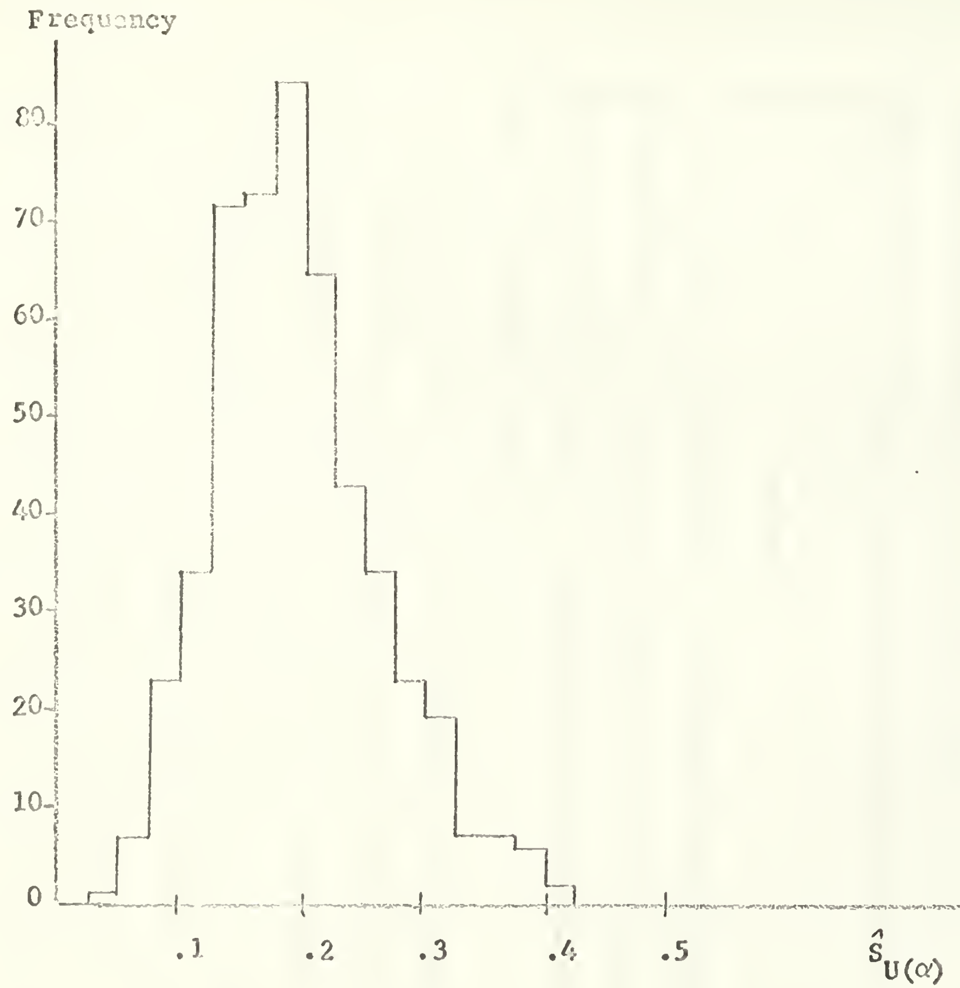
$-\ln R_S = .3239$

$A = .2839$

$\bar{x} = .4154$

FIGURE 2

HISTOGRAM OF DISTRIBUTION FOR CASE 3, α = .20



$$\hat{S}_{U(\alpha)} = \hat{S} + K_{\alpha}\hat{\sigma}$$

i	n_i
1	250
2	40
3	120
4	15
5	130
6	65
7	70
8	130
9	30
10	20
11	75
12	90
13	100
14	60
15	60

$p_i = .99, \quad i = 1, \cdots, 15$

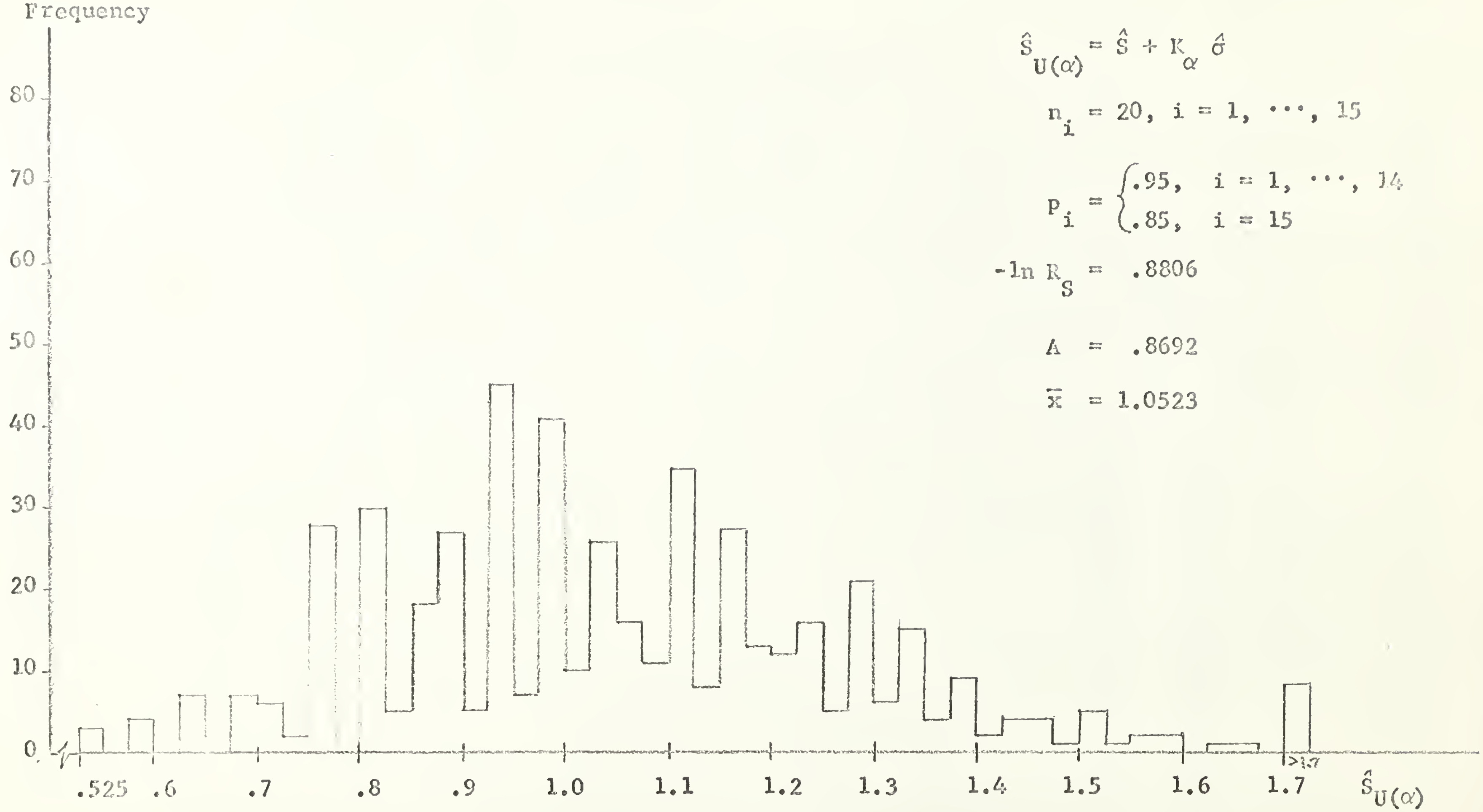
$-\ln R_S = .1508$

$\Lambda = .1354$

$\ddot{x} = .1951$

FIGURE 3

HISTOGRAM OF DISTRIBUTION FOR CASE 4, $\alpha = .20$



$$\hat{S}_{U(\alpha)} = \hat{S} + K_\alpha \hat{\sigma}$$

$$n_i = 20, \; i = 1, \cdots, 15$$

$$p_i = \begin{cases} .95, & i = 1, \cdots, 14 \\ .85, & i = 15 \end{cases}$$

$$-\ln R_S = .8806$$

$$A = .8692$$

$$\bar{x} = 1.0523$$

FIGURE 4

HISTOGRAM OF DISTRIBUTION FOR CASE 5, $\alpha = .20$

further to the right on the $\hat{S}_{U(\alpha)}$ axis than the previous distributions, its lowest value being .5251. This distribution also shows large dispersion about its mean, with a standard deviation of .232.

From these form illustrative cases, coupled with an analysis of the other 26 cases which are not graphed, some general conclusions as to the shape of the distribution can be made. As the p_i decrease in value, the range of the distribution increases and the distribution shifts to the right on the $\hat{S}_{U(\alpha)}$ abscissa. If the p_i are varied instead of constant, the dispersion of the distribution about its mean is larger. If the n_i are varied, the distribution can take on a larger number of discrete values within its range, and the distribution becomes unimodal.

RESULTS

The results of the 30 cases, ten different combinations of the parameters k, n_i, and p_i for each of three values of α, are presented in Table I below. In Table II, the results are compared with the Woods-Borsting Method.

Column (1) of Table I is self-explanatory. The headings of the rest of the table have been previously defined above, but are reiterated here for use in referring to the table. Column (2) is the number of components in the system. Column (3) is the size of the sample tested for each component of the system. Column (4) is the probability of success for each component of the system. Column (5) is the system reliability

TABLE I

RESULTS OF COMPUTER SIMULATION

(1) CASE	INPUT PARAMETERS (2) k	(3) n_i	(4) p_i	(5) R_S	(6) α	(7) $\hat{S}+K_\alpha\hat{\sigma}$, mean	(8) $\hat{S}+K_\alpha\hat{\sigma}$, variance	(9) exp(-A)	(10) $\lvert exp(-A)-R_S \rvert$
1	15	20	.995, i=1,...,14 .85, i=15	.7924	.05	.4060	.0249	.8761	.0837
					.10	.3680	.0219	.8264	.0340
					.20	.3220	.0185	.7976	.0052
2	15	20	.999 .962 .998 .970 .965 .980 .973 .990 .967 .992 .978 .985 .975 .850 .969	.6289	.05	.7057	.0384	.6480	.0191
					.10	.6514	.0348	.6332	.0043
					.20	.5857	.0307	.6600	.0311
3	13	150 19 90 5 75 125 100 63 125 125 18 59 28	.995 .900 .985 .980 .979 .995 .988 .970 .982 .995 .980 .968 .967	.7233	.05	.5032	.0354	.7729	.0496
					.10	.4635	.0296	.7630	.0397
					.20	.4154	.0234	.7524	.0291
4	15	250 30 40 20 120 75 15 90 130 100 65 60 70 60 130	.99, i=1,...,15	.8601	.05	.2369	.0068	.8891	.0290
					.10	.2180	.0058	.8780	.0179
					.20	.1951	.0047	.8734	.0133
5	15	20	.95, i=1,...,14 .85, i=15	.4145	.05	1.2193	.0634	.4321	.0176
					.10	1.1438	.0589	.4314	.0169
					.20	1.0523	.0538	.4193	.0048

TABLE I

(continued)

(1) CASE	INPUT PARAMETERS (2) k	(3) n_i	(4) p_i	(5) R_S	(6) α	(7) $\hat{S}+K_\alpha\hat{\sigma}$, mean	(8) $\hat{S}+K_\alpha\hat{\sigma}$, variance	(9) exp(-A)	(10) $\lvert\exp(-A)-R_S\rvert$
6	15	20	.995, i=1,...,14 .85, i=15	.7924	.05	.3942	.0233	.8761	.0837
					.10	.3570	.0204	.8264	.0340
					.20	.3118	.0172	.8002	.0078
7	15	250 30 40 20 12 75 15 90 130 100 65 60 70 60 130	.99, i=1,...,15	.8601	.05	.2319	.0083	.8935	.0334
					.10	.2125	.0070	.8891	.0290
					.20	.1889	.0055	.8811	.0210
8	15	50	.995, i=1,...,14 .85, i=15	.7924	.05	.3356	.0072	.8156	.0232
					.10	.3115	.0066	.8082	.0158
					.20	.2822	.0058	.8068	.0144
9	15	20, i=1,...,4 150, i=5 20, i=6,..,15	.995, i=1,...,14 .85, i=15	.7924	.05	.4123	.0220	.8697	.0773
					.10	.3741	.0193	.8233	.0309
					.20	.3279	.0162	.7976	.0052
10	15	20, i=1,..,14 150, i=15	.995, i=1,...,14 .85, i=15	.7924	.05	.3398	.0098	.8250	.0326
					.10	.3165	.0084	.8200	.0276
					.20	.2881	.0068	.8044	.0120

based on the input parameters of columns (2) through (4). Column (6) is the value of α which divides each case into three parts. The mean and variance of the distribution of $\hat{S} + K_{\alpha} \hat{\sigma}$ are shown in columns (7) and (8). Column (9) is the exponential of the α^{th} percentile of the distribution, and column (10) is the difference between columns (9) and (5), the measure of accuracy of the procedure.

In Table I, cases 1 and 6 are identical except that the number of failures, f_i, were generated by the computer using different entry points to the random number generator subroutine. Comparing the values of $|\exp(-A) - R_S|$ for these two cases, it is seen that they are the same for $\alpha = .05$ and $\alpha = .10$, and differ only in the third decimal place for $\alpha = .20$. Therefore, it appears that 500 replications of each case is sufficient to overcome random fluctuations in the numbers generated, and it was to investigate this area that these two cases were chosen identically. In reading Table I, one should compare cases, 1, 6, 9, and 10, as 9 and 10 differ from 1 and 6 only in choice of one of the n_i; and 9 and 10 differ from each other only in the p_i with which this one different n_i is associated. Case 8 is also comparable to the above four cases, the difference here being that the n_i were increased from 20 to 50 for all i, which appreciably increases the accuracy of the procedure. The results of cases 4 and 7 should also be compared as they differ only in the choice of n_3, and again it is seen that the larger the n_i, the better the accuracy, all other things being equal.

To see the effect of the p_i on the accuracy of the procedure, consider cases 3, 5, 6. In case 3 the average p_i, call it $\overline{p}_i$ = .9702; for case 5, $\overline{p}_i$ = .9433, and in case 6, $\overline{p}_i$ = .9853. At the α = .05 level, the accuracy improves as $\overline{p}_i$ decreases, which is as it should be. As the p_i decrease the q_i increase, and thus when the random number generator subroutine computes f_i, the number of failures, there exists a greater number of discrete values f_i can take on, hence a correspondingly greater number of values $\hat{S} + K_\alpha \hat{\sigma}$ can take on in the lower tail of the distribution. It is therefore more likely that the 5^{th} percentile of the distribution will be close to R_S. In general, the worst results are for the distributions in which α = .05 due to the sparsity of different discrete values the distribution can take on.

In general, the results were not too satisfactory. In only five of the 30 cases was exp(-A) within .01 of the true system reliability, R_S. Of considerable importance is the fact that in every case the difference exp(-A)-R_S was positive indicating that the values of A, the α^{th} percentile of the distribution, were too small. This led to the addition of a continuity correction factor, developed in Section 4, to compensate for the fact that $\hat{S}$, in reality a discrete random variable, was fitted by a continuous probability distribution.

Table II compares the above results with the results of the Woods-Borsting Method. Columns (1), (2), (3), (5) and (7) were used in Table I, and are identical in definition here. Column

TABLE II

COMPARISON OF WOODS-BORSTING METHOD AND COMPUTER SIMULATION

(1) CASE	(2) R_S	(3) α	(4) $R_{S,\alpha}$	(5) exp(-A)	(6) $R_{S,\alpha}-R_S$	(7) $\lvert exp(-A)-R_S \rvert$
1	.79	.05	.69	.88	.10	.09
		.10	.71	.83	.08	.04
		.20	.77	.80	.02	.01
2	.63	.05	.56	.65	.07	.02
		.10	.61	.63	.02	.00
		.20	.63	.66	.00	.03
3	.72	.05	.71	.77	.01	.05
		.10	.69	.76	.03	.04
		.20	.71	.75	.01	.03
4	.86	.05	.87	.89	.01	.03
		.10	.87	.88	.01	.02
		.20	.87	.87	.01	.01
5	.41	.05	.38	.43	.03	.02
		.10	.40	.43	.01	.02
		.20	.42	.42	.01	.01
6	.79	.05	.69	.88	.10	.09
		.10	.71	.83	.08	.04
		.20	.77	.80	.02	.01
7	.86	.05	.80	.89	.06	.03
		.10	.86	.89	.00	.03
		.20	.87	.88	.01	.02
8	.79	.05	.77	.82	.02	.03
		.10	.78	.81	.01	.02
		.20	.79	.81	.00	.02
9	.79	.05	.69	.87	.10	.08
		.10	.74	.82	.05	.03
		.20	.77	.80	.02	.10
10	.79	.05	.82	.83	.03	.04
		.10	.82	.82	.03	.03
		.20	.80	.80	.01	.01

(4) is the α^{th} percentile of the distribution $\hat{R}_S$ of [1], and column (6) is a measure of the accuracy of the Woods-Borsting Method. Table II does not include all the cases simulated in [1], rather only those which are identical to the cases of Table I. As mentioned in the Introduction, the average error in the Woods-Borsting Method was .036 for all cases simulated. The average of column (6) is .032 while the average of column (7) of Table II is .03, so at least based on these average error figures the fitting of a Normal distribution rather than a Gamma, to $\hat{S}$ appears to be slightly better, but not significantly.

The cases of principal interest are those where component sample sizes differ, since this is the problem the procedure was designed to cope with. These are cases 3, 4, 7, 9, and 10. The average error for these cases under the normal distribution assumption (column (7)) is also .03, so it appears to be no better or no worse for varying sample sizes than for constant sample sizes, based upon the few cases presented in Table II.

4. SIMULATED DISTRIBUTION OF CONFIDENCE LIMIT WITH CONTINUITY CORRECTION FACTOR, AND RESULTS

This section is devoted to a heuristic development of the continuity correction factor mentioned above, a presentation in Table III of the results of the same ten cases with the correction factor included, and a comparison of these results with the results of the Woods-Borsting Method.

DEVELOPMENT OF CONTINUITY CORRECTION FACTOR

Since $\hat{S}$ is a random variable able only to take on discrete values, but was assumed, in (29), to be a normal random variable, a continuity correction was deemed necessary. Consider (8), in which (9) through (11) have been substituted

$$\hat{T}_i = \frac{2n_i - 3}{2(n_i-1)} \cdot \frac{f_i}{n_i} + \frac{n_i}{2(n_i-1)} \cdot \frac{f_i^2}{n_i^2}$$

$$= \frac{(2n_i - 3)\, f_i + f_i^2}{2n_i\,(n_i-1)} \qquad (45)$$

For every possible number of failures, $0 \leq f_i \leq n_i$, $\hat{T}_i$ assumes a different value, and if these values were plotted on the real line, the size of the interval between adjacent values would vary due to the nature of $\hat{T}_i$. It was felt that, just as in the Normal approximation to the Binomial (where the continuity correction factor is $\frac{1}{2n}$, if n is the sample size), the continuity correction factor should be related to the size

of the interval. Toward this end define

$$\hat{T}_i' \equiv \frac{(2n_i-3)}{2(n_i-1)} \cdot \frac{(f_i+1)}{n_i} + \frac{n_i}{2(n_i-1)} \cdot \frac{(f_i+1)^2}{n_i^2} \tag{46}$$

$$= \frac{f_i^2 + (2n_i-1)\, f_i + 2(n_i-1)}{2n_i(n_i-1)}$$

Then the size of the interval between the two adjacent values is

$$\hat{T}_i' - T_i \tag{47}$$

and if the size of this interval is normalized by dividing by the sample size, n_i, and arbitrarily take ½ the result, then

$$\frac{1}{2} \cdot \frac{1}{n_i} (\hat{T}_i' - \hat{T}_i)$$

$$= \frac{1}{2} \cdot \frac{1}{n_i} \quad \frac{\{f_i^2+(2n_i-1)f_i+2(n_i-1)\} - \{(2n_i-3)f_i+f_i^2\}}{2n_i(n_i-1)}$$

$$= \frac{2n_i + 2f_i - 2}{4n_i^2\,(n_i-1)} \tag{48}$$

$$= \frac{n_i + f_i - 1}{2n_i^2\,(n_i-1)}$$

is the continuity correction factor added. Notice that when $f_i = 0$, which it will a large majority of the time because $q_i \ll 1$, the continuity correction factor reduces to

$$\frac{1}{2n_i^2} \tag{49}$$

Define

$$T_i^* = T_i + \frac{T_i - T_i}{2n_i} \tag{50}$$

then it follows that

$$\hat{S}^* = \sum_{i=1}^{k} \hat{T}_i^* \tag{51}$$

$$\hat{\sigma}^* = \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i^*}{n_i}} \tag{52}$$

which may be combined in a probability statement similar to (34a)

$$1 - \alpha \doteq P\,[S \le \hat{S}^* + K_\alpha\, \hat{\sigma}^*] \tag{52a}$$

$$= P\,[S \le \hat{S}^*_{U(\alpha)}\,]$$

and the new lower $100(1 - \alpha)\%$ confidence interval, comparable to (36), is,

$$1 - \alpha \doteq P\left[R_S \ge \exp\left(-\sum_{i=1}^{k} \hat{T}_i^* - K_\alpha \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i^*}{n_i}}\right)\right] \tag{53}$$

and associated lower confidence limit is

$$R^*_{S,L(\alpha)} = \exp\left(- \sum_{i=1}^{k} \hat{T}_i^* - K_\alpha \sqrt{\sum_{i=1}^{k} \frac{\hat{T}_i^*}{n_i}} \right) \tag{54}$$

the distribution of which is simulated as before for exactly the same ten cases, and the results are given in Table III.

DISCUSSION OF SHAPE OF DISTRIBUTION WITH CONTINUITY CORRECTION ADDED

Figures 5 through 8 below are identical to figures 1 through 4, except that the simulated distributions plotted have the continuity correction factor added. There are no significant differences apparent; all four of the distributions with the correction factor added have essentially the same shape as their respective predecessors. All of the general comments made with respect to the first four figures also apply to these four, and they are presented for comparison purposes only.

RESULTS OF ADDING THE CONTINUITY CORRECTION FACTOR

The results of the same 30 cases, but with the continuity correction factor added, are presented in Table III below. The results are compared with the Woods-Borsting Method in Table IV, and with the results before the continuity correction factor was added in Table I.

Table III is almost identical to Table I, the only differences being in columns (7) and (8), which are the mean and

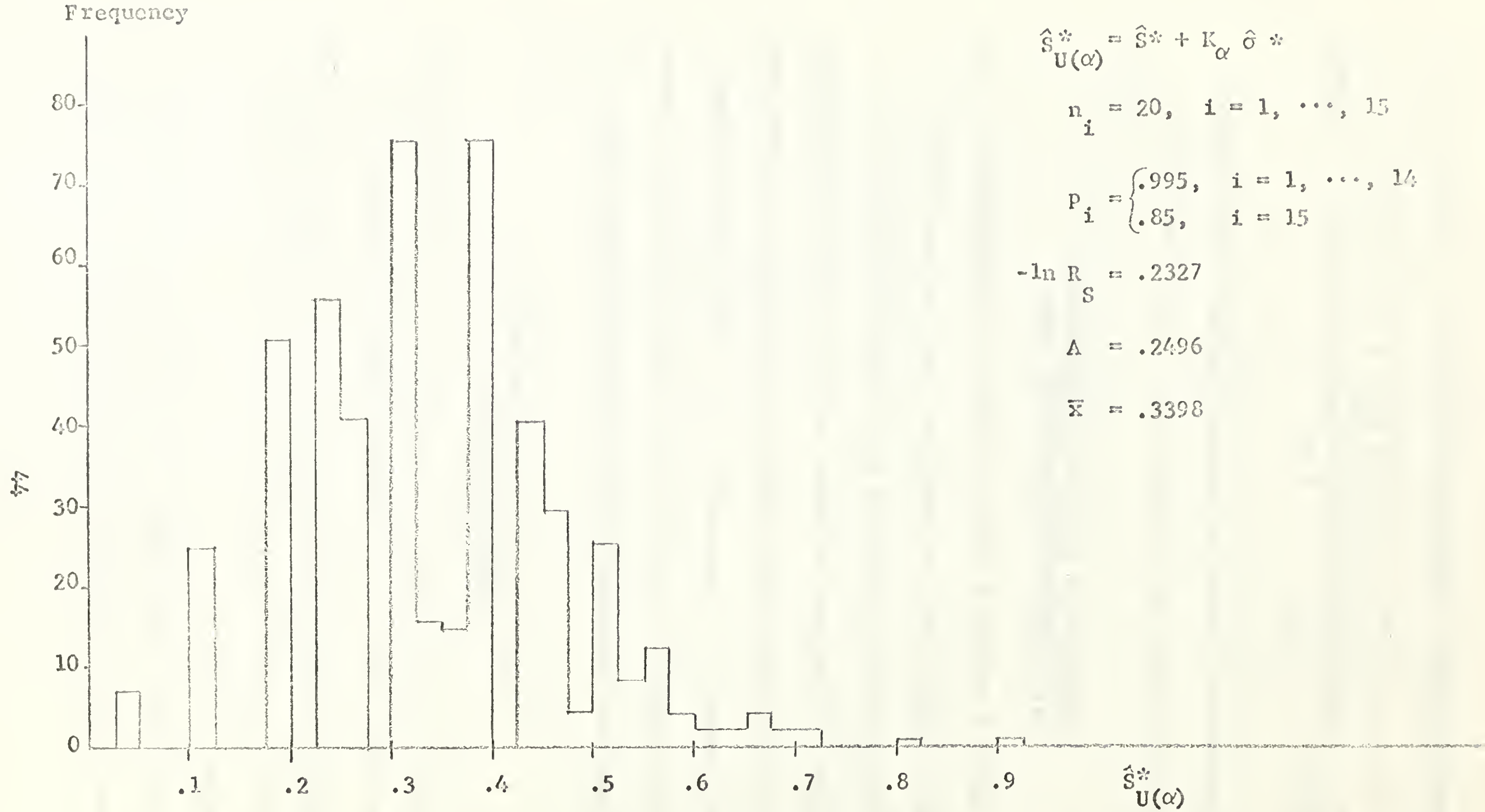
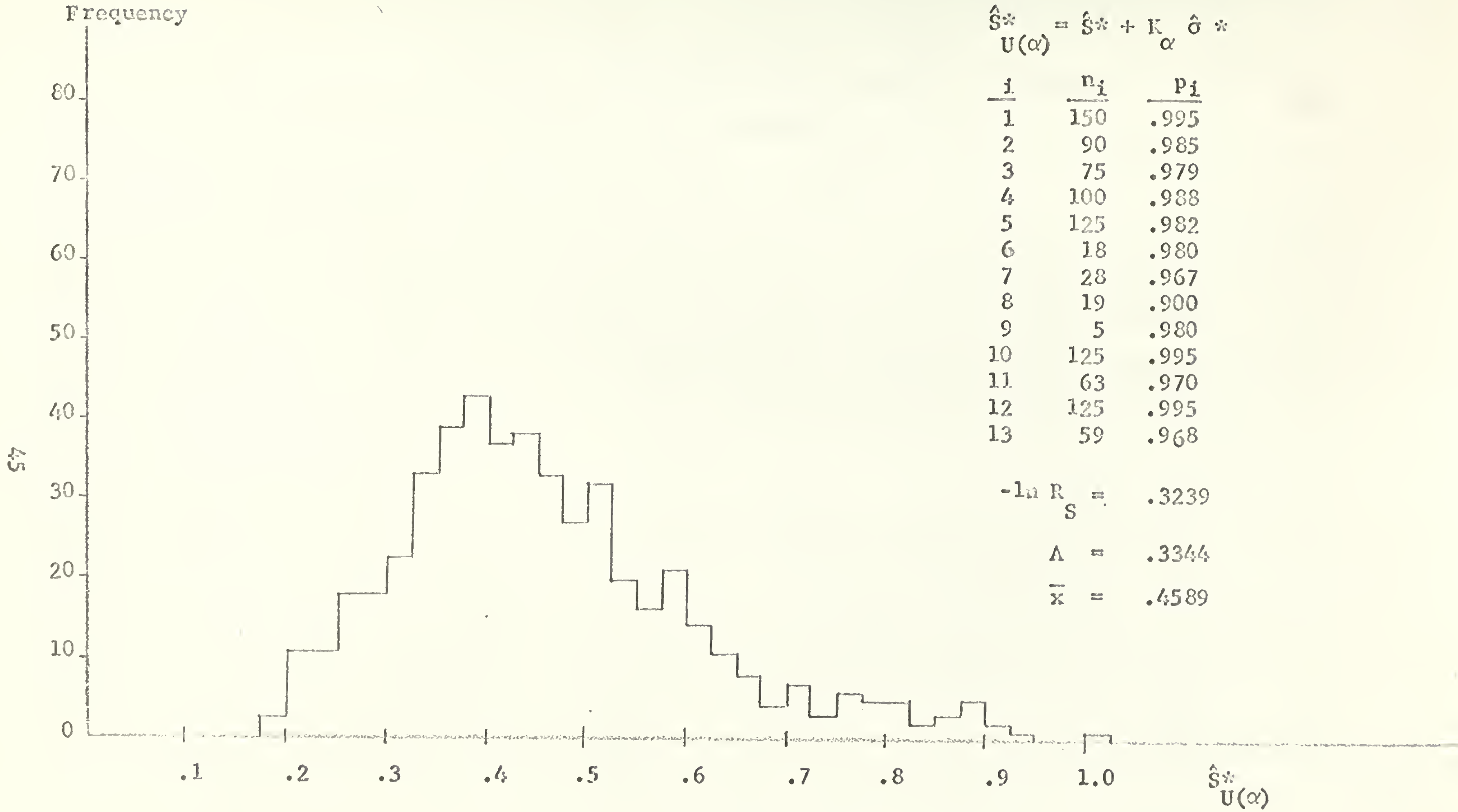


FIGURE 5

HISTOGRAM OF DISTRIBUTION WITH CONTINUITY CORRECTION ADDED FOR CASE 1, $\alpha = .20$



$$\hat{S}^*_{U(\alpha)} = \hat{S}^* + K_\alpha \hat{\sigma}^*$$

i	n_i	p_i
1	150	.995
2	90	.985
3	75	.979
4	100	.988
5	125	.982
6	18	.980
7	28	.967
8	19	.900
9	5	.980
10	125	.995
11	63	.970
12	125	.995
13	59	.968

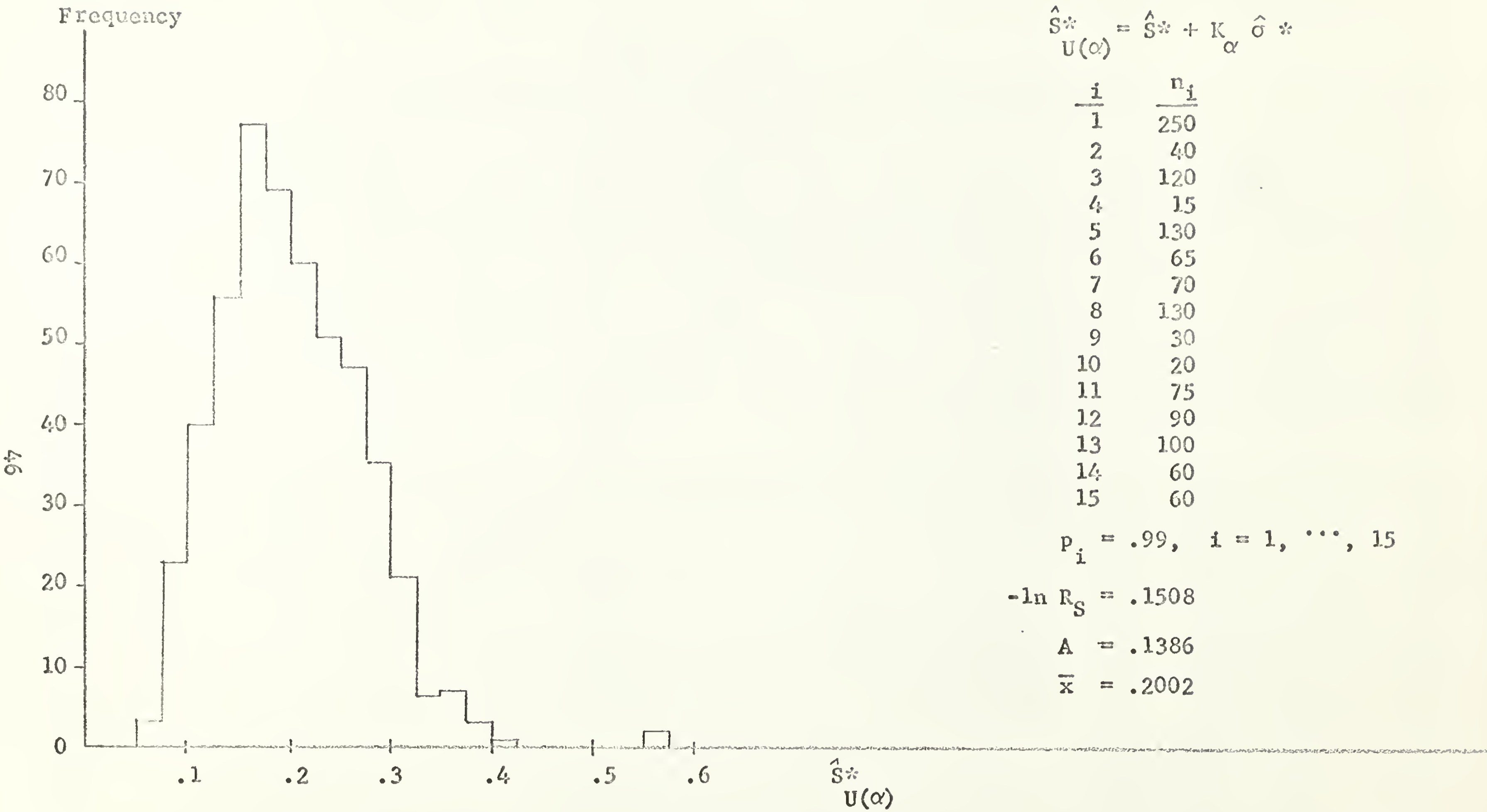
$-\ln R_S = .3239$

$A = .3344$

$\bar{x} = .4589$

FIGURE 6

HISTOGRAM OF DISTRIBUTION WITH CONTINUITY CORRECTION ADDED FOR CASE 3, $\alpha = .20$



$$\hat{S}^*_{U(\alpha)} = \hat{S}^* + K_\alpha \hat{\sigma}^*$$

i	n_i
1	250
2	40
3	120
4	15
5	130
6	65
7	70
8	130
9	30
10	20
11	75
12	90
13	100
14	60
15	60

$$p_i = .99, \quad i = 1, \cdots, 15$$

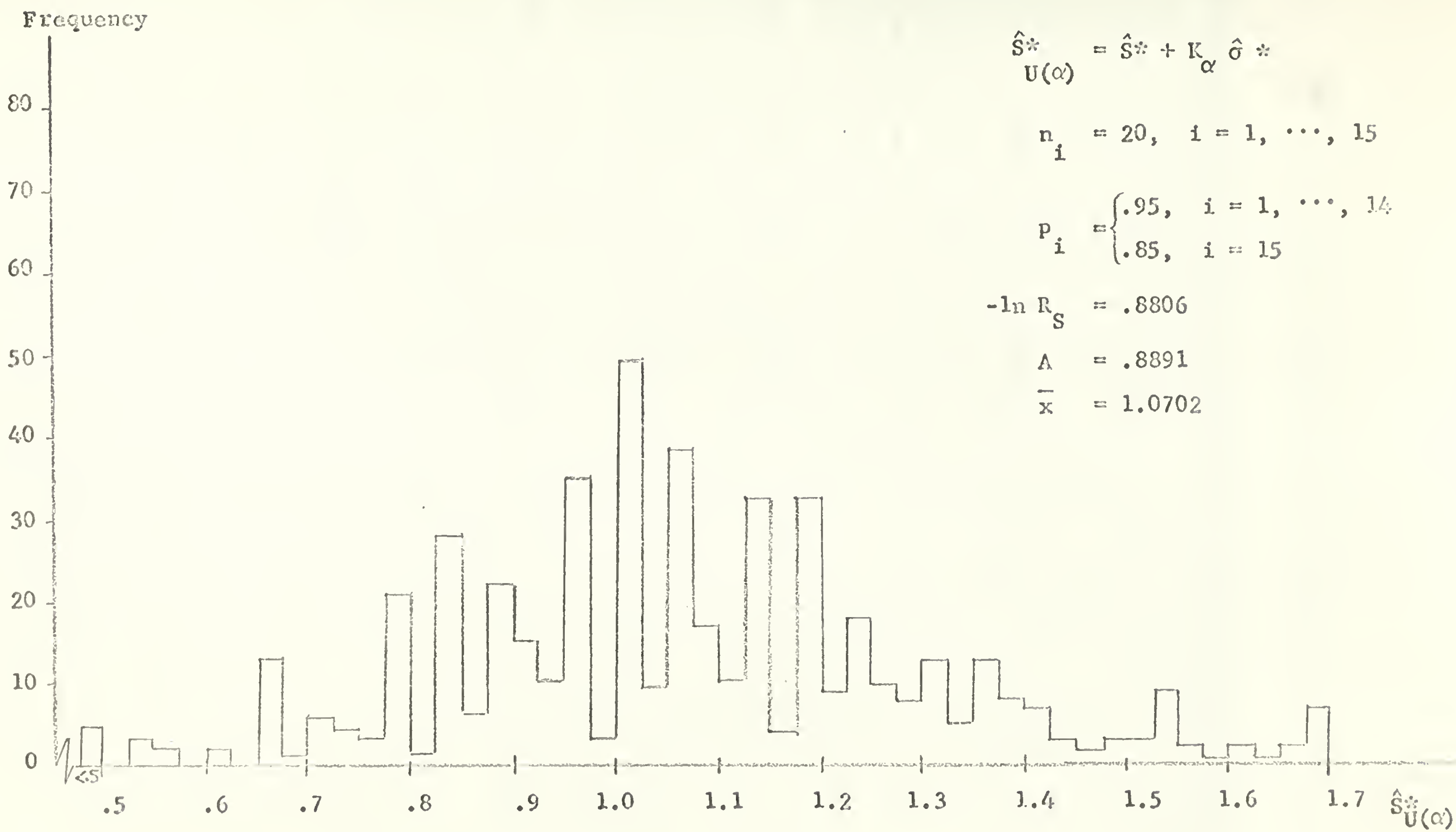
$$-\ln R_S = .1508$$

$$A = .1386$$

$$\bar{x} = .2002$$

FIGURE 7

HISTOGRAM OF DISTRIBUTION WITH CONTINUITY CORRECTION ADDED FOR CASE 4, α = .20



$$\hat{S}^*_{U(\alpha)} = \hat{S}^* + K_\alpha \hat{\sigma}^*$$

$$n_i = 20, \quad i = 1, \cdots, 15$$

$$p_i = \begin{cases} .95, & i = 1, \cdots, 14 \\ .85, & i = 15 \end{cases}$$

$$-\ln R_S = .8806$$

$$A = .8891$$

$$\bar{x} = 1.0702$$

FIGURE 8

HISTOGRAM OF DISTRIBUTION WITH CONTINUITY CORRECTION
ADDED FOR CASE 5, $\alpha = .20$

TABLE III

RESULTS OF COMPUTER SIMULATION WITH CONTINUITY CORRECTION FACTOR

(1)	INPUT PARAMETERS			(5)	(6)	(7)	(8)	(9)	(10)
	(2)	(3)	(4)			$\hat{S}^* + K_\alpha \hat{\sigma}^*$,	$\hat{S}^* + K_\alpha \hat{\sigma}^*$,	$\exp(-\hat{A}^*)$	
CASE	k	n_i	p_i	R_S	α	mean	variance		$\lvert \exp(-\hat{A}^*) - R_S \rvert$
1	15	20	.995 i=1,...,14 .85,i=15	.7924	.05	.4269	.0230	.8476	.0552
					.10	.3876	.0203	.8043	.0119
					.20	.3398	.0173	.7791	.0133
2	15	20	.999 .962 .998 .970 .965 .980 .973 .990 .967 .992 .978 .985 .975 .850 .969	.6289	.05	.7316	.0394	.6738	.0449
					.10	.6761	.0358	.6519	.0230
					.20	.6089	.0317	.6415	.0126
3	13	150 19 90 5 75 125 100 63 125 125 18 59 28	.995 .900 .985 .980 .979 .995 .988 .970 .982 .995 .980 .968 .967	.7233	.05	.5622	.0327	.7248	.0015
					.10	.5155	.0279	.7218	.0015
					.20	.4589	.0226	.7158	.0075
4	15	250 30 40 20 120 75 15 90 130 100 65 60 70 60 130	.99, i=1,...,15	.8601	.05	.2437	.0066	.8815	.0214
					.10	.2240	.0057	.8763	.0162
					.20	.2002	.0047	.8702	.0101
5	15	20	.95, i=1,...,14 .85,i=15	.4145	.05	1.2386	.0678	.4459	.0314
					.10	1.1625	.0630	.4429	.0084
					120	1.0702	.0573	.4102	.0043

TABLE III

(continued)

(1) CASE	INPUT PARAMETERS (2) k	(3) n_i	(4) p_i	(5) R_S	(6) α	(7) $\hat{S}^* + K_\alpha \hat{\sigma}_*^*$ mean	(8) $\hat{S}^* + K_\alpha \hat{\sigma}_*^*$ variance	(9) $\exp(-\hat{A}^*)$	(10) $\lvert\exp(-\hat{A}^*) - R_S\rvert$
6	15	20	.995, i=1,...,14 .85, i=15	.7924	.05	.4326	.0255	.8476	.0552
					.10	.3929	.0225	.8043	.0119
					.20	.3449	.0192	.7817	.0107
7	15	250 30 40 20 12 75 15 90 130 100 65 60 70 60 130	.99, i=1,...,15	.8601	.05	.2598	.0091	.8786	.0185
					.10	.2374	.0076	.8753	.0152
					.20	.2101	.0061	.8710	.0109
8	15	50	.995, i=1,...,14 .85,i=15	.7924	.05	.3444	.0076	.8123	.0199
					.10	.3199	.0069	.8064	.0140
					.20	.2902	.0062	.8040	.0116
9	15	20, i=1,...,4 150, i=5 20, i=6,..,15	.995, i=1,...,14 .85, i=15	.7924	.05	.4265	.0239	.8371	.0447
					.10	.3875	.0211	.8028	.0104
					.20	.3401	.0180	.7828	.0096
10	15	20, i=1,..,14 150, i=15	.995, i=1,...,14 .85, i=15	.7924	.05	.3634	.0091	.8017	.0093
					.10	.3377	.0079	.8006	.0082
					.20	.3065	.0065	.7916	.0008

variance of the new simulated distributions with continuity correction included, and in columns (9) and (10) where A* has been used to denote the α^{th} percentile of the distribution with correction factor included. Cases 1 and 6, which are identical except for the entry point into the random number generator, are, as in Table I, identical in column (10) for $\alpha = .05$ and .10, and differ only in the third decimal place for $\alpha = .20$. Cases 1, 6, 8,9, and 10 are very similar and should be compared to each other in reading the Table. Cases 4 and 7 are also comparable to each other. Cases 3, 5, 6 show the affect of the p_i on the accuracy of the procedure.

The overall picture presented by Table III is much more encouraging than that of Table I. In nine of the 30 cases the measure of accuracy (column (8)) was less than .01 and in two more was less than .0105, and considering the number of significant figures in the input parameters, the results are probably not significant to four decimal places, so these two could be taken as equal to .01, giving 11 cases less than or equal to .01. The average difference in magnitude between e^{-A*} and R_S was .017. For cases 3, 4, 7, 9, 10, the ones in which component sample sizes vary, the average difference was .012, a slight improvement.

Table IV compares the results with the continuity correction factor to the results of the Woods-Borsting Method. This Table is identical with Table II with the exception of columns (5) and (7), where A* has been used, vice A, to denote the α^{th} percentile

TABLE IV

COMPARISON OF WOODS-BORSTING METHOD AND COMPUTER SIMULATION WITH CONTINUITY CORRECTION ADDED

(1) CASE	(2) R_S	(3) α	(4) $R_{S,\alpha}$	(5) $\exp(-A^*)$	(6) $R_{S,\alpha}-R_S$	(7) $\exp(-A^*)-R_S$
1	.79	.05	.69	.85	.10	.06
		.10	.71	.80	.08	.01
		.20	.77	.78	.02	.01
2	.63	.05	.56	.67	.07	.04
		.10	.61	.65	.02	.02
		.20	.63	.64	.00	.01
3	.72	.05	.71	.72	.01	.00
		.10	.69	.72	.03	.00
		.20	.71	.72	.01	.00
4	.86	.05	.87	.88	.01	.02
		.10	.87	.88	.01	.02
		.20	.87	.87	.01	.01
5	.41	.05	.38	.45	.03	.04
		.10	.40	.42	.01	.01
		.20	.42	.41	.01	.00
6	.79	.05	.69	.85	.10	.06
		.10	.71	.80	.08	.01
		.20	.77	.76	.02	.01
7	.86	.05	.80	.88	.06	.02
		.10	.86	.88	.00	.02
		.20	.87	.87	.01	.01
8	.79	.05	.77	.81	.02	.02
		.10	.78	.80	.01	.01
		.20	.79	.80	.00	.01
9	.79	.05	.69	.84	.10	.05
		.10	.74	.80	.05	.01
		.20	.77	.78	.02	.01
10	.79	.05	.82	.80	.03	.01
		.10	.82	.80	.03	.01
		.20	.80	.79	.01	.00

of the new distribution with correction factor added. As in Table II, the entries in the body of the Table have been rounded to two significant figures due to the nature of the input parameters. The average error in column (7) with the continuity correction factor added is .017, and is significantly less than the average error of column (6), .036, for the Woods-Borsting Method. However, it is noteworthy that this high average is attributable to three large values. Taking away the three largest values from both column (6) and (7), the column averages become .023 and .021, respectively, which are insignificantly different.

In Table V the results obtained before the continuity correction factor was added are compared to the results obtained upon introducing the correction factor. The body of the table gives the number of cases for which the magnitude of the measure of accuracy was within the range specified in column (1). The sizes of intervals in columns (1) were so chosen because of the fact that $|R_S - \exp(-A)|$ is only accurate to two places, even though computed to four places. For example, the four-place value .0142, when rounded off becomes .01, hence the choice of the second interval of .0051 to .0150. The addition of the continuity correction factor made a marked improvement in the results as can be seen both in the average value of the accuracy measure, .028 before, compared to .017 after, and in the increased number of cases that fall in the intervals representing the best accuracy. Before the continuity correction was added

TABLE V

COMPARISON OF COMPUTER SIMULATIONS WITH AND WITHOUT CONTINUITY CORRECTION FACTOR

(1) $\lvert R_S - \exp(-A) \rvert$	(2) without CC	(3) with CC
.0000 -- .0050	2	4
.0051 -- .0150	6	15
.0151 -- .0250	7	6
.0251 -- .0350	10	1
.0351 -- .0450	1	2
.0451 -- .0550	1	0
.0551 -- .0650	0	2
.0651 -- .0750	0	0
.0751 -- .0850	3	0
> .0850	0	0
TOTAL	30	30
Average Accuracy $\lvert R_S - \exp(-A) \rvert$	.028	.017

only one-half the cases were within .025 of true system reliability, while after the addition of this correction, 25 of the 30 cases had accuracy better than .025. Additionally, prior to the use of the correction factor, there were three cases in which the accuracy was no better than .075, whereas upon adding the correction factor, there were no cases worse than .06. Based on these 30 cases it is amply apparent that the continuity correction factor improves the procedure considerably, and considering the fact that (52a) is an approximate confidence interval, this procedure appears usable in situations where rough lower bounds on system reliability are permissible.

5. CONCLUSIONS AND ACKNOWLEDGEMENTS

In this section the results of [1] and the results of this thesis are compiled together in Table VI, comparisons are made and results discussed. Secondly, general conclusions based on the entire thesis and recommendations for further investigation are made.

TABULATION OF OVERALL RESULTS

In Table VI are presented the results of computer simulations of four different approaches to the problem of obtaining a lower $100(1 - \alpha)\%$ confidence interval on system reliability. Two of the four are from [1], the well-known Poisson approximation and the Woods-Borsting Method. The remaining two are the ones developed in this thesis, fitting a normal distribution to S, both with and without a continuity correction factor.

Column (4), Y_α, is the α^{th} percentile of the simulated distribution of a confidence limit involving a Chi-square random variable and based on the Poisson approximation to the Binomial. Inherent in this procedure the fact that sample sizes of all components must be equal, hence no entries can be made for cases 3, 4, 7, 9, and 10. The rest of the column headings have been explained above, either in connection with Table II or Table IV. If all four methods were perfectly accurate, all the entries in columns (4) through (7) would exactly equal column (2).

Notice how poor the Poisson approximation is in Case 5, a case where the p_i are relatively small (see Table I), and in

TABLE VI

COMPARISON OF COMPUTER SIMULATIONS WITH AND WITHOUT CONTINUITY CORRECTION FACTOR, POISSON APPROXIMATION, AND WOOD-BORSTING METHOD

(1) CASE	(2) R_S	(3) α	(4) Y_α	(5) $R_{S,\alpha}$	(6) exp(-A)	(7) exp(-A*)
1	.79	.05	.76	.69	.88	.85
		.10	.73	.71	.83	.80
		.20	.72	.77	.80	.78
2	.63	.05	.54	.56	.65	.67
		.10	.54	.61	.63	.65
		.20	.55	.63	.66	.64
3	.72	.05	---	.71	.77	.72
		.10	---	.69	.76	.72
		.20	---	.71	.75	.72
4	.86	.05	---	.87	.89	.88
		.10	---	.87	.88	.88
		.20	---	.87	.87	.87
5	.41	.05	.09	.38	.43	.45
		.10	.11	.40	.43	.42
		.20	.15	.42	.42	.41
6	.79	.05	.76	.69	.88	.85
		.10	.73	.71	.83	.80
		.20	.72	.77	.80	.78
7	.86	.05	---	.86	.89	.88
		.10	---	.86	.89	.88
		.20	---	.87	.88	.87
8	.79	.05	.76	.77	.82	.81
		.10	.76	.78	.81	.81
		.20	.77	.79	.81	.80
9	.79	.05	---	.69	.87	.84
		.10	---	.74	.82	.80
		.20	---	.77	.80	.78
10	.79	.05	---	.82	.83	.80
		.10	---	.82	.82	.80
		.20	---	.80	.80	.79

case 2. The procedures recorded in columns (5) through (7) also have places where accuracy is bad, but these occur for the most part in cases where $\alpha = .05$, which is to be expected, for the reasons mentioned in Section 3. From Table VI one can see that these three procedures display much more consistency in their accuracy than does the Poisson approximation method. The average error of the Poisson approximation is .049, and reiterating, the average errors for columns (5) through (7) are .036, .028, and .017 respectively.

SUMMARY

In the preceding three sections, an approximate lower $100(1 - \alpha)\%$ confidence interval on system reliability has been developed, based on the procedure of [1], but differing from [1] in that the probability distribution fitted to $\hat{S}$ by the method of moments is normal rather than Gamma. The lower confidence limit of this confidence interval is a random variable, and in order to test the accuracy of the approximate confidence interval, the distribution of this confidence limit was simulated by digital computer, and its α^{th} percentile compared to true system reliability. Figures 1 through 4 are histograms of four representative distributions; the reasons for their shapes are discussed. The level of accuracy obtained was not satisfactory, and hence a continuity correction factor was developed and added to the lower confidence limit. The distribution was again simulated by computer, the histograms of the same four distributions were plotted, and the α^{th} percentiles of all 30

cases were compared to true system reliability as a measure of accuracy of the procedure. The changes in shape of the four distributions plotted were negligible.

In conclusion, it appears that the method developed above by fitting a normal distribution to $\hat{S}$ without continuity correction factor is no better or no worse than the method developed in [1] by fitting a Gamma distribution to $\hat{S}$. Both have several large errors in accuracy, and both have average accuracy error on the order of .03. After the continuity correction factor is added the method of this thesis is certainly better than the same method before the continuity correction factor was added, as can be seen from Table V, and appears to be better, on the basis of average accuracy, than the method of [1]. However, in examination of Table IV, cases 4 and 8 have better accuracy by the Woods-Borsting Method, so one must conclude that neither of the methods is universally better than the other. It is not realistic to compare the method of [1] to the method developed above with continuity correction factor, because no correction factor has been added to the former, a point mentioned by, and currently under investigation by the authors of [1]. Another conclusion that came as a by-product of this investigation is that one should use the Poisson approximation with great skepticism as the results are evidently not at all consistent in their accuracy.

It appears that further investigation of this entire procedure is called for, and some recommendations as to the

paths this investigation should take are mentioned below. First, no variation in k, the number of components of the system, was made, and this is certainly a cogent area for exploration to see what effects this parameter has on the accuracy. Secondly, it would seem desirable to simulate a great many cases, with different combinations of the parameters k, p_i, n_i, with an eye toward devising a set of rules by which one could decide in advance which method would give the best results, dependent upon the parameters involved. It is impossible to make any statements of this kind with only ten different combinations of parameters to examine. A third area for further investigation is one already mentioned in Section 2, that of investigating the error involved in truncation of the expression for the variance of S.

I would like to thank Dr. W. Max Woods of the Operations Analysis Department of the U. S. Naval Postgraduate School for suggesting this problem to me, for allowing me to use the procedure of [1] as a starting point, and for the helpful suggestions given me throughout the course of this investigation. I would like to also acknowledge the help of Dr. Rex H. Shudde, also of the Operations Analysis Department, for allowing me to use the computer program he developed for the procedure of [1], to which only minor modifications were necessary, and for the assistance in programming he gave me.

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APPENDIX I

DERIVATION OF $\hat{T}_i$, THE UNBIASED ESTIMATOR OF T_i

In order that $\hat{T}_i$ be an unbiased estimator of T_i, it is necessary that

$$T_i = E[\hat{T}_i]$$

Rewriting the above expression, using (6) and (8)

$$q_i + \frac{q_i^2}{2} = E\left[\frac{a_i f_i}{n_i} + \frac{b_i f_i^2}{2}\right]$$

and, substituting (11)

$$q_i + \frac{q_i^2}{2} = E\left[\frac{a_i f_i}{n_i} + \frac{b_i f_i^2}{2n_i^2}\right]$$

$$= \frac{a_i}{n_i} E\,[f_i] + \frac{b_i}{2n_i^2} E\,[f_i^2]$$

$$= \frac{a_i n_i q_i}{n_i} + \frac{b_i}{2n_i^2} (n_i^2 q_i^2 - n_i q_i^2 + n_i q_i)$$

$$= a_i q_i + \frac{b_i q_i^2}{2} (1 - \frac{1}{n_i}) + b_i q_i$$

$$= (a_i + \frac{b_i}{2n_i})\, q_i + (b_i - \frac{b_i}{n_i})\, \frac{q_i^2}{2}$$

Then equating coefficients of q_i and $\frac{q_i^2}{2}$

$$a_i + \frac{b_i}{2n_i} = 1$$

$$b_i - \frac{b_i}{n_i} = 1$$

and solving simultaneously

$$a_i + \frac{b_i}{2n_i} = 1$$

$$\frac{1}{2}(b_i) - \frac{1}{2}\left(\frac{b_i}{n_i}\right) = \frac{1}{2}(1)$$

$$\Rightarrow \qquad a_i + \frac{b_i}{2} = \frac{3}{2}$$

$$\Rightarrow \qquad a_i = \frac{1}{2}(3-b_i)$$

Substituting back

$$\frac{1}{2}(3-b_i) + \frac{b_i}{2n_i} = 1$$

$$3 - b_i + \frac{b_i}{n_i} = 2$$

$$\Rightarrow \qquad b_i = \frac{n_i}{n_i-1}$$

and again substituting

$$a_i = \frac{1}{2}\left(3 - \left(\frac{n_i}{n_i-1}\right)\right)$$

$$= \frac{2n_i - 3}{2(n_i-1)}$$

Thus,

$$\hat{T}_i = \frac{2n_i - 3}{2(n_i - 1)} \hat{q}_i + \frac{n_i}{n_i - 1} \frac{\hat{q}_i^{\,2}}{2}$$

is an unbiased estimator of T_i.

APPENDIX II

COMPUTATION OF VARIANCE OF $\hat{S}$

$$\mathrm{Var}(\hat{S}) = \mathrm{Var}\left(\sum_{i=1}^{k} \hat{T}_i\right) = \sum_{i=1}^{k} \mathrm{Var}\,(\hat{T}_i)$$

$$\mathrm{Var}(\hat{T}_i) = E[\hat{T}_i^{\,2}] - E^2[\hat{T}_i]$$

$$= E[\hat{T}_i^{\,2}] - T_i^{\,2}$$

$$E[\hat{T}_i^{\,2}] = E[(a_i\hat{q}_i + \frac{b_i\hat{q}_i^{\,2}}{2})^2]$$

$$= E[a_i^{\,2}\hat{q}_i^{\,2} + a_ib_i\hat{q}_i^{\,3} + \frac{b_i^{\,2}\hat{q}_i^{\,4}}{4}]$$

$$= \frac{a_i^{\,2}}{n_i^{\,2}}\,E\,[f_i^{\,2}] + \frac{a_ib_i}{n_i^{\,3}}\,E\,[f_i^{\,3}] + \frac{b_i^{\,2}}{4n_i^{\,4}}\,E[f_i^{\,4}]$$

From [4]

$$E[f_i^{\,2}] = n_i^{\,2}q_i^{\,2} - n_iq_i^{\,2} + n_iq_i$$

$$E[f_i^{\,3}] = E[(f_i - E[f_i])^3] + 3E[f_i]E[f_i^{\,2}] - 2E^3[f_i]$$

$$= n_i^{\,3}q_i^{\,3} - 3n_i^{\,2}q_i^{\,3} + 2n_iq_i^{\,3} + 3n_i^{\,2}q_i^{\,2} - 3n_iq_i^{\,2} + n_iq_i$$

$$E[f_i^4] = E[(f_i - E[f_i])^4] + 4E[f_i]E[f_i^3] - 6E^2[f_i]\,E[f_i^2] + 3E^4[f_i]$$

$$= n_i^4 q_i^4 - 6n_i^3 q_i^3 + 11n_i^2 q_i^4 - 18n_i^2 q_i^3 + 7n_i^2 q_i^2 - 6n_i q_i^4 + 12n_i q_i^3 - 7n_i q_i^2 + n_i q_i$$

Therefore

$$E[\hat{T}_i^2] = \frac{(2n_i-3)^2}{2^2(n_i-1)^2 n_i^2}\left\{n_i^2 q_i^2 - n_i q_i^2 + n_i q_i\right\}$$

$$+ \frac{(2n_i-3)\,n_i}{2(n_i-1)^2 n_i^3}\left\{n_i^3 q_i^3 - 3n_i^2 q_i^3 + 2n_i q_i^3 + 3n_i^2 q_i^2 - 3n_i q_i^2 + n_i q_i\right\}$$

$$+ \frac{n_i}{4(n_i-1)^2 n_i^4}\left\{n_i^4 q_i^4 - 6n_i^3 q_i^4 + 6n_i^3 q_i^3 + 11n_i^2 q_i^4 - 18n_i^2 q_i^3 + 7n_i^2 q_i^2 - 6n_i q_i^4 + 12n_i q_i^3 - 7n_i q_i^2 + n_i q_i\right\}$$

$$E[\hat{T}_i^2] = \frac{(4n_i^2 - 12n_i + 9)\ (n_i q_i^2 - q_i^2 + q_i) + (4n_i - 6)(n_i^2 q_i^3 - 3n_i q_i^3}{4(n_i - 1)^2 n_i}$$

$$\frac{+ 2q_i^3 + 3n_i q_i^2 - 3q_i^2 + q_i)}{4(n_i - 1)^2 n_i}$$

$$+ \frac{(n_i^3 q_i^4 - 6n_i^2 q_i^4 + 6n_i^2 q_i^3 + 11n_i q_i^4 - 18n_i q_i^3 + 7n_i q_i^2}{4(n_i - 1)^2 n_i}$$

$$\frac{- 6q_i^4 + 12q_i^3 - 7q_i^2 + q_i)}{4(n_i - 1)^2 n_i}$$

$$= \frac{n_i^3 (q_i^4 + 4q_i^3 + 4q_i^2) + n_i^2 (-6q_i^4 - 12q_i^3 - 4q_i^2 + 4q_i)}{4n_i (n_i - 1)^2}$$

$$\frac{+ \ n_i (11q_i^4 + 8q_i^3 - 2q_i^2 - 8q_i)}{4n_i (n_i - 1)^2}$$

$$+ \frac{(-6q_i^4 + 2q_i^2 + 4q_i)}{4n_i (n_i - 1)^2}$$

Let

$$A = q_i^4 + 4q_i^3 + 4q_i^2$$

$$B = -6q_i^4 - 12q_i^3 - 4q_i^2 + 4q_i$$

$$C = 11q_i^4 + 8q_i^3 - 2q_i^2 - 8q_i$$

$$D = -6q_i^4 + 2q_i^2 + 4q_i$$

Then

$$E[\hat{T}_i^{\,2}] = \frac{An_i^3 + Bn_i^2 + Cn_i + D}{4n_i^3 - 8n_i^2 + 4n_i}$$

$$= \frac{A}{4} + \frac{B + 2A}{4n_i} + \frac{(C + 2B + 3A)n_i + (D - B - 2A)}{4n_i(n_i - 1)^2}$$

$$= T_i^2 + \frac{-q_i^4 - q_i^3 + q_i^2 + q_i}{n_i} + \frac{(2q_i^4 - 4q_i^3 + 2q_i^2)n_i}{4n_i(n_i-1)^2}$$

$$\frac{-(2q_i^4 - 4q_i^3 + 2q_i^2)}{4n_i(n_i-1)^2}$$

$$= T_i^2 + \frac{q_i}{n_i} + \frac{q_i^2}{2n_i} + \frac{q_i^2}{2n_i} - \frac{(q_i^3 + q_i^4)}{n_i} + \frac{q_i^2(q_i-1)^2}{2n_i(n_i-1)}$$

$$E[\hat{T}_i^2] - T_i^2 = Var(\hat{T}_i) = \frac{T_i}{n_i} + \frac{q_i^2}{n_i}\left\{\frac{1}{2} - (q_i + q_i^2) + \frac{(q_i-1)^2}{2(n_i-1)}\right\}$$

which, when simplified, gives

$$Var(\hat{T}_i) = \frac{T_i}{n_i} + \frac{q_i^2}{n_i}\left\{\frac{3q_i^2 - n_i\ (2q_i^2 + 2q_i - 1)}{2(n_i-1)}\right\}$$

$$= \frac{T_i}{n_i} + \frac{q_i^2}{2(n_i-1)}\left\{1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i\right\}$$

which we approximate by

$$Var(\hat{T}_i) \doteq \frac{T_i}{n_i}$$

$$Var(\hat{S}) \doteq \sum_{i=1}^{k} \frac{T_i}{n_i}$$

and the truncation error is

$$\sum_{i=1}^{k}\left(\frac{q_i^2}{2(n_i-1)}\left\{1 + \frac{3q_i^2}{n_i} - 2q_i^2 - 2q_i\right\}\right)$$

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1. ORIGINATING ACTIVITY (Corporate author)

U. S. Naval Postgraduate School
Monterey, California

2a. REPORT SECURITY CLASSIFICATION

UNCLASSIFIED

2b. GROUP

----

3. REPORT TITLE

AN INVESTIGATION OF CONFIDENCE LIMITS ON SYSTEM RELIABILITY WHEN COMPONENT SAMPLE SIZES VARY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

----

5. AUTHOR(S) (Last name, first name, initial)

Isaacson, Robert T., LT, USN

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1966	72	4

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

----

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

----

10. AVAILABILITY/LIMITATION NOTICES

[redacted]

11. SUPPLEMENTARY NOTES

----

12. SPONSORING MILITARY ACTIVITY

U. S. Naval Postgraduate School
Monterey, California

13. ABSTRACT

The method of [1] for obtaining an approximate lower 100(1-α)% confidence limit on System Reliability is used, except, instead of a Gamma probability distribution, a Normal probability distribution is used as the underlying distribution for $\hat{S}$, the estimator of the negative logarithm of system reliability.

An investigation is made of the errors involved in the truncations and approximations used throughout the development. A continuity correction factor is developed, and confidence limits on system reliability, resulting from computer simulation, are presented both with and without the inclusion of this continuity correction factor.

14. KEY WORDS	LINK A		LINK B		LINK C	
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System Reliability Confidence Limits Continuity Correction Factor Computer Simulation of Probability Distribution Unequal Component Sample Sizes						

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